Name of college — S. S. College, J-bad
Deptt — Mathematics
Topic — Problem based on Curvature (Cartesian Curves) (Differential Equation)
Period — I + II
Time — 10:15 AM - 11:15 AM + 11 AM - 11:45 AM
Class — B.Sc (Hons)
Date — 27-07-2020
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Study material → 1. If Equation of curve is

$$s = f(\psi)$$

then $\rho = \dfrac{ds}{d\psi}$

2. When equation of curve is $y = f(x)$

then $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$ where $y_1 = \dfrac{dy}{dx} = p$

$y_2 = \dfrac{d^2y}{dx^2} = q$

3. When Equation of curve is

$x = f(t) \quad y = g(t)$

then $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

4. When the equation of Curve is $f(x,y) = 0$

then $\rho = \dfrac{(f_x^2 + f_y^2)^{3/2}}{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}$

Problem 1: → for the curve in which

$s = ce^{x/c}$ show that

$$c\rho = s\sqrt{s^2 - c^2}$$

Solution - Since Here equation of curve is of the form $s = f(\psi)$

then $\rho = \dfrac{ds}{d\psi}$

$$s = ce^{x/c}$$

$$\frac{ds}{dx} = c\, e^{x/c} \cdot \frac{1}{c} = e^{x/c}$$

$$\because s = ce^{x/c}$$

$$\Rightarrow \frac{s}{c} = e^{x/c}$$

$$\therefore \frac{ds}{dx} = \frac{s}{c} \qquad \left[\because \frac{dx}{ds} = \cos\psi \Rightarrow \frac{ds}{dx} = \sec\psi, \quad \frac{ds}{d\psi} = a\sec\psi, \quad \frac{dy}{dx} = \tan\psi \right]$$

$$\Rightarrow \sec\psi = \frac{s}{c}$$

$$\therefore s = c\sec\psi$$

$$\frac{ds}{d\psi} = c\,\frac{d\sec\psi}{d\psi}$$

$$= c\sec\psi\tan\theta$$

$$= c\sec\psi\sqrt{\sec^2\psi - 1}$$

As $\frac{s}{c} = \sec\psi$

$$= c \cdot \frac{s}{c}\sqrt{\frac{s^2}{c^2} - 1}$$

$$= \frac{s\sqrt{s^2 - c^2}}{c}$$

But $\frac{ds}{d\psi} = \rho$

$$\therefore \rho = \frac{s\sqrt{s^2 - c^2}}{c}$$

$$\Rightarrow c\rho = s\sqrt{s^2 - c^2}$$

proved

Problem 2:

Find the radius of curvature at any point of the parabola $y^2 = 4ax$ and hence deduce that the radius of curvature at the vertex of parabola is equal to semi latus rectum.

Here Equation of curve is of the form $y = f(x)$

this radius of curvature

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$$

Here Equation of curve is $y^2 = 4ax$

Diff. with respect to x,

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Diff. again with respect to x

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx}$$

$$= -\frac{2a}{y^2} \times \frac{2a}{y} = -\frac{4a^2}{y^3}$$

Here $y_1 = \frac{2a}{y}$ $y_2 = -\frac{4a^2}{y^3}$

$$\therefore \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$= -\frac{[4a^2+y^2]^{3/2}}{\frac{4a^2}{y^3} \times y^3} = -\frac{[4a^2+4ax]^{3/2}}{4a^2}$$

$$= -\frac{[4a(a+x)]^{3/2}}{4a^2} = -\frac{4^{3/2}a^{3/2}(a+x)^{3/2}}{4a^2}$$

$$\therefore \rho = \frac{-8a^{3/2}(a+x)^{3/2}}{4a^2}$$

$$= -2a^{-1/2}(a+x)^{3/2}$$

Magnitude of

$$\rho = 2a^{-1/2}(a+x)^{3}$$

Now radius of curvature at vertex at $(0,0)$

$$\rho = 2a^{-1/2}(a+x)^{3/2}$$

$$\rho^2 = 4a^{-1}(a+x)^3$$

put $x=0, y=0$

$$\rho^2 = 4 a^{-1} a^3$$

$$= 4a^2$$

$\therefore \rho = 2a$ But length of Latus rectum $= 4a$

$2\rho = 4a =$ length of Latus rectum

$$\therefore \rho = \frac{\text{latus rectum}}{2}$$

Problem 3. Find the radius of curvature at any point $(a\cos t, b\sin t)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Here equation of ellipse is

$x = a\cos t \quad y = b\sin t$

∴ radius of curvature

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} \quad \text{———— (1)}$$

$x' = \frac{dx}{dt} = -a\sin t \qquad y' = \frac{dy}{dt} = b\cos t$

$y'^2 = b^2\cos^2 t$

$x'^2 = a^2\sin^2 t$

$y'' = -b\sin t$

$x'' = -a\cos t$

putting all these values in (1)

$$\rho = \frac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{(-a\sin t)(-b\sin t) - (-a\cos t)(b\cos t)}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab(\sin^2 t + \cos^2 t)}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

Problem 4 – In the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Prove that –

$\rho = \frac{a^2b^2}{p^3}$ Where p = Central perpendicular on the tangent.

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Here $x = a\cos t$ $\quad y = b\sin t$

$x' = -a\sin t$ $\quad y' = b\cos t$

$x'' = -a\cos t$ $\quad y'' = -b\,\cancel{\cos t}\,\sin t$

Now
$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} \quad \text{——— } (1)$$

$x'^2 = a\sin^2 t$ $\quad y'^2 = b^2\cos^2 t$

$x'y'' = ab\sin^2 t\,\cancel{\cos t}$ $\quad x''y' = -ab\cos^2 t$

$$\therefore \rho = \frac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{ab\sin^2 t - (-ab\cos^2 t)}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

Now the equation of tangent at

$(x_1 = a\cos t \quad y_1 = b\sin t)$

is $\dfrac{x\,a\cos t}{a^2} + \dfrac{y\,b\sin t}{b^2} = 1$

$\Rightarrow x\dfrac{\cos t}{a} + y\cdot\dfrac{\sin t}{b} - 1 = 0$

$$p^2 = \frac{1}{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}} = \frac{a^2b^2}{a^2\sin^2 t + b^2\cos^2 t}$$

$$\rho = \left(\frac{a^2b^2}{p^2}\right)^{3/2}\cdot\frac{1}{ab} = \frac{a^3b^3}{p^3}\cdot\frac{1}{ab} = \frac{a^2b^2}{p^3}$$